Oefening 7 (cycloïde)

$$
\begin{aligned}
x &= R(\theta - \sin\theta) \\
y &= R(1 - \cos\theta)
\end{aligned}
$$

- Lengte

$$
\begin{aligned}
\dot{x} &= R(1 - \cos\theta) \\
\dot{y} &= R\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\, d\theta \\
&= R\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\
&= R\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\
&\Downarrow \quad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}
\end{aligned}
$$

$$
\begin{aligned}
L &= 2R\int_0^{2\pi} \sin\frac{\theta}{2} d\theta \\
&= 2R\cdot\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} = 8R
\end{aligned}
$$

- Oppervlakte

$$
\begin{aligned}
S(x') &= \int_0^{x'} y(x)\,dx \\
S(\theta') &= \int_0^{\theta'} y(\theta)\,\frac{dx}{d\theta}d\theta \\
&\Downarrow \\
S &= R^2\int_0^{2\pi}(1-\cos\theta)(1-\cos\theta)\,d\theta \\
&= R^2\int_0^{2\pi}\left(1-2\cos\theta+\cos^2\theta\right)d\theta \\
&= 2\pi R^2 - 2R^2[\sin\theta]_0^{2\pi} + R^2\int_0^{2\pi}\cos^2\theta d\theta \\
&\Downarrow \quad \cos^2\theta = \frac{\cos 2\theta + 1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi R^2 + R^2 \int_0^{2\pi} \frac{\cos 2\theta + 1}{2} d\theta \\
&= 2\pi R^2 + R^2 \left(\frac{1}{4} [\sin 2\theta]_0^{2\pi} + \frac{2\pi}{2} \right) \\
&= 3\pi R^2
\end{aligned}
$$

- Kromtestraal

$$
\mathcal{R} = \pm \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\ddot{y}\dot{x} - \ddot{x}\dot{y}} \quad \text{teken zodat } \mathcal{R} > 0
$$

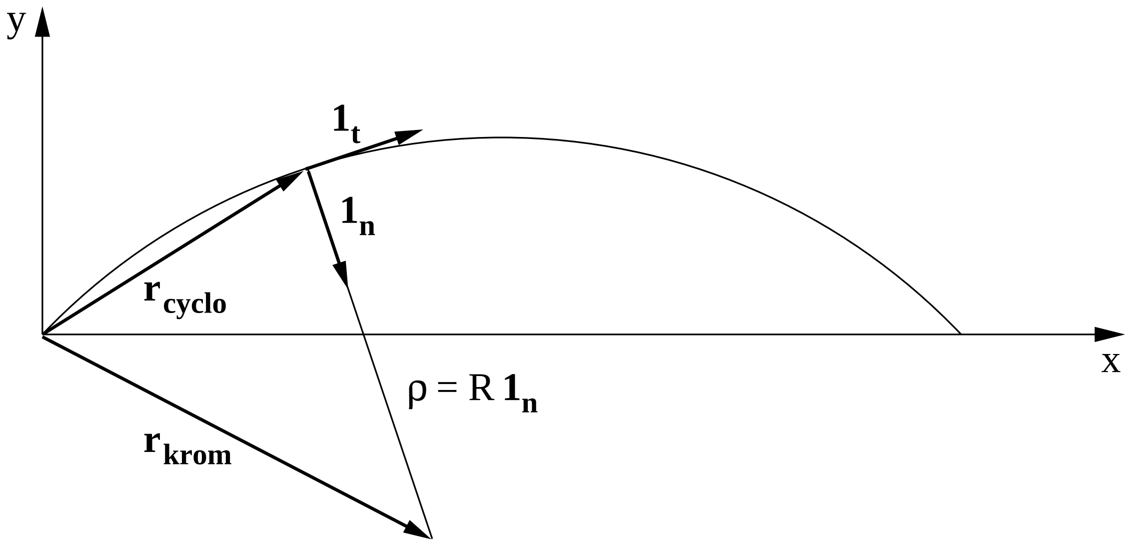
$$
\begin{aligned}
\dot{x}^2 + \dot{y}^2 &= R^2 \left(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right) \\
&= R^2 (2 - 2\cos\theta) \\
&= 4R^2 \sin^2 \frac{\theta}{2}
\end{aligned}
$$

$$
\begin{aligned}
\ddot{x} &= R\sin\theta \\
\ddot{y} &= R\cos\theta \\
&\Downarrow
\end{aligned}
$$

$$
\begin{aligned}
\ddot{y}\dot{x} - \ddot{x}\dot{y} &= R^2 \cos\theta\,(1-\cos\theta) - R^2\sin^2\theta \\
&= R^2\,(\cos\theta - 1) \\
&= -2R^2\sin^2\frac{\theta}{2}
\end{aligned}
$$

$$
\Rightarrow \mathcal{R} = \frac{8R^3\sin^3\frac{\theta}{2}}{2R^2\sin^2\frac{\theta}{2}} = 4R\sin\frac{\theta}{2}
$$

- Meetkundige plaats v.d. krommingsmiddelpunten



$$
\begin{aligned}
\mathbf{r_{krom}} &= \mathbf{r_{cyclo}} + \rho \\
&= \mathbf{r_{cyclo}} + \mathcal{R}\mathbf{1_n}
\end{aligned}
$$

Richting van de raaklijn aan de cycloïde:

$$(\dot{x},\dot{y}) = (R(1-\cos\theta), R\sin\theta)$$

Vector $\mathbf{V}_\perp = (X,Y)$ loodrecht op deze richting bepalen:

$$\begin{aligned}
(X,Y)\cdot(R(1-\cos\theta), R\sin\theta) &= 0 \\
&\Downarrow \\
XR(1-\cos\theta) + YR\sin\theta &= 0 \\
&\Downarrow \\
X &= \frac{-Y\sin\theta}{(1-\cos\theta)}
\end{aligned}$$

Als we $X = \sin\theta$ kiezen vinden we $Y = \cos\theta - 1$, of

$$\begin{aligned}
\mathbf{V}_\perp &= (\sin\theta, \cos\theta - 1) \\
&\Downarrow \\
|\mathbf{V}_\perp| &= \sqrt{\sin^2\theta + \cos^2\theta - 2\cos\theta + 1} \\
&= \sqrt{2-2\cos\theta} = 2\sin\frac{\theta}{2} \\
&\Downarrow
\end{aligned}$$

$$
\begin{aligned}
\mathbf{1_n} &= \frac{\mathbf{V}_\perp}{|\mathbf{V}_\perp|} \\
&= \left(\frac{\sin\theta}{2\sin\frac{\theta}{2}}, \frac{\cos\theta - 1}{2\sin\frac{\theta}{2}}\right)
\end{aligned}
$$

Dus:

$$
\begin{aligned}
\mathbf{r_{krom}} &= \mathbf{r_{cyclo}} + \mathcal{R}\mathbf{1_n} \\
&= \left(R\left(\theta - \sin\theta\right), R\left(1 - \cos\theta\right)\right) \\
&\quad + 4R\sin\frac{\theta}{2}\left(\frac{\sin\theta}{2\sin\frac{\theta}{2}}, \frac{\cos\theta - 1}{2\sin\frac{\theta}{2}}\right) \\
&= \left(R\left(\theta + \sin\theta\right), R\left(\cos\theta - 1\right)\right)
\end{aligned}
$$