PART A

THEORY

A.0.
INTRODUCTION

Since experimental modal analysis is a synergy of several engineering disciplines, it appeals to a wide theoretical background. The aim of part A of this book is to summarise all these theoretical aspects. This summary tries to focus on these aspects of the theory that are directly related to experimental modal analysis.

An experimental modal analysis consists of five phases. The first phase is building the test *set-up*: suspending the test object, attaching transducers, connecting the data acquisition system, calibration, ... The second phase is the *acquisition of data* and, most often, the estimation of frequency response functions. The third phase is a *system identification* phase: the determination of the vibration characteristics of the system from the measured input-output data. The fourth phase is the *validation* of the obtained results. All these phases are necessary in order to reach the fifth phase: using the obtained information for improving the system in a systematic way. All these steps are based on a theoretical background:

- The first chapter will derive from basic vibration theory the modal analysis theory: it is the overall theoretical foundation of experimental and analytical modal analysis. The theory for single degree of freedom systems defines the concepts of system poles, *resonance frequency*, *damping ratio*, *residues*. These concepts are extended to the multiple degree of freedom case, yielding concepts such as *mode shapes*, *modal participation factors*, *modal mass*, *modal scaling*. It treats the generally viscously damped, proportionally damped and undamped case
- The second chapter covers the theory behind the data acquisition phase: (digital) signal processing and the frequency response function estimation procedures. It first discusses the *Fourier transform* (and its properties) as a means of transforming data from time to frequency domain and vice versa. It illustrates the most important errors in digital signal processing: *aliasing* and *leakage*. Finally it shows how the *frequency response function* can be estimated in different ways from auto- and crosspower spectra.
- The third chapter discusses briefly the theory of several parameter identification methods. These methods estimate the modal parameters, system poles (resonance frequencies and damping factors), mode shapes and modal participation factors. These modal parameters describe the vibration behaviour of the system. It distinguishes between *single degree of freedom* and *multiple degree of freedom* methods, between *single input* or *multiple input* methods, between methods yielding *local* or *global estimates* of the modal parameters, between *time* or *frequency domain* methods, ...

- The fourth chapter contains some mathematical tools for *evaluation and validation* of the obtained modal model of the system. These techniques help to judge if the results of the next step will be reliable or not.
- Chapter five shows how measured modal parameters can be used for prediction purposes, with techniques as *sensitivity analysis*, *dynamic coupling* of structures, *system modification* prediction (chapter five). The ultimate aim of all these techniques is the improvement of the dynamic behaviour of an existing structure, already in use or in a prototype or design stage.
- Chapter six shows how finite element modelling and experimental modal analysis can be linked, either for finite element *model improvement*, either for *optimising the test* set-up.

Part B. will cover more practical aspects of experimental modal analysis: instrumentation, calibration and test set-up, excitation, parameter estimation in practice, error detection, prevention and reduction.

A.1. ANALYTICAL AND EXPERIMENTAL MODAL ANALYSIS

A.1.0. INTRODUCTION

In order to better understand many of the practicalities of experimental modal analysis a good understanding of the basic theory is needed. The aim of this chapter on *analytical and experimental modal analysis* is to offer the engineer a firm theoretical foundation for investigating the dynamics of structures. Basically, two approaches exist to study the vibrations of a system.

The first one is analytical: starting from the knowledge of the structure geometry, the boundary conditions and material characteristics the mass, stiffness and damping distribution of the structure is expressed in terms of mass, stiffness and damping matrices. These contain sufficient information to determine the system modal parameters (natural frequencies, damping factors and mode shapes). The theory will show that these modal parameters completely describe the dynamics of the system.

The second approach starts from measurements of dynamic input forces and output responses on (a prototype of) the structure of interest. These measurements are most often transformed into frequency response functions, i.e. the ratio between output and input as a function of frequency. The theory will show that these frequency response functions can be expressed in terms of the modal parameters. Hence, the second step of an experimental modal analysis consists of estimating these parameters from the measured frequency response functions.

The specific aim of this chapter is to show the relation between both approaches. The theory will first cover the single degree of freedom case, basically to introduce concepts as: system poles, natural frequencies, damping ratios, residues. This basic development is then extended for multiple degree of freedom systems. This part will introduce more concepts: modal vectors, modal coordinates, orthogonality, modal mass, stiffness and damping, modal vector scaling, ...

Before starting with the theory, the concept of degree of freedom (DOF) is explained and the basic assumptions of modal analysis are discussed.

The degrees of freedom of a rigid body mass are the minimum number of coordinates needed to locate this mass in space. The number of degrees of freedom for such case is six: three translation degrees of freedom (x, y, z) to locate the mass center of gravity and three rotational degrees of freedom ($\theta_x, \theta_y, \theta_z$) to define its orientation. Since every deformable structure can be considered as a combination of an infinite number of (small) rigid body masses, all structures have an infinite number of degrees of freedom. However, all these structures will be approximated as a combination of a limited number of such masses (a set of physical points of interest, with six degrees of freedom each), yielding a finite number of degrees of freedom, N. This number also defines the dimensions of the analytical mass, stiffness and damping matrices and the number of theoretically present natural frequencies and mode shapes. However, practical limitations will limit the number of measured degrees of freedom: rotational degrees of freedom are very difficult to measure and the bounded frequency range limits the number of detectable modes. Hence, the analytical model may have N degrees of freedom, the experiments will offer information about N_i input degrees of freedom, N_o output degrees of freedom and N_m detectable mode shapes.

The basic assumptions for the modal analysis theory are:

linearity: the structure dynamic behaviour is linear: the output of any combination of inputs is equal to the same combination of the respective outputs. The dynamics can be presented by a set of linear, second order differential equations. Each modal analysis test should start with a check of the linear dynamic behaviour of the structure (section B.2.3.1).

time invariance: the structure dynamic characteristics don't change in time. Hence, the coefficients of the differential equations are constants, invariable with respect to time. A typical time invariance problem may occur due to the mass loading of the motion transducers that have to be attached to the structure (section B.1.2.2).

observability: this means that all necessary data to determine the system dynamic characteristics of interest can be measured. In order to avoid problems with observability a proper choice of response degrees of freedom is important (sections A.6.2.3 and B.2.2.3)

Most often is assumed that the structure obeys *Maxwell's reciprocity* principle: the response in a point p due to an input in point q, equals the response in point q due to the same input at point p. This assumption yields symmetric mass, stiffness, damping and frequency response function matrices. Multiple input testing allows to check the reciprocity of the measured frequency response functions (section B.2.3.1).

Although the information in this chapter is general vibration information it is primarily based upon reference a.1.1, a course text developed by R.Allemang and others. Some additional information is found in reference a.1.2 by D.Ewins.

A.1.1. SINGLE DEGREE OF FREEDOM SYSTEMS

This section will introduce several basic concepts with regard to modal analysis. These concepts will reappear in the discussion of multiple degree of freedom systems (section A.1.2). Starting from the force equilibrium equation, this section will derive an expression for the *transfer function*. This is the ratio of the output over the input as a

function of the Laplace variable. The denominator of this function is the system characteristic equation, defining the ***system poles***, ***natural frequencies*** and ***damping ratios***. The numerators of the partial fraction expansion of the transfer function contain the ***residues***. Section A.1.1.5 discusses the frequency and time domain equivalents of the transfer function, i.e. the ***frequency response function*** and the ***impulse response function***. At last, section A.1.1.6 shows the effects of a mass, stiffness and damping change on the dynamic behaviour of a single degree of freedom system.

A.1.1.1. System equation, transfer function

The force equilibrium for a viscously damped single degree of freedom (SDOF) system (fig. a.1.1) expresses the balance between inertial, damping, elastic and external forces:

(a.1.1) $$M\ddot{x}(t)+C\dot{x}(t)+Kx(t)=f(t)\,.$$

where: M: mass
C: damping
K: stiffness
$\ddot{x},\dot{x},x$: acceleration, velocity, displacement
f: external force
t: time variable

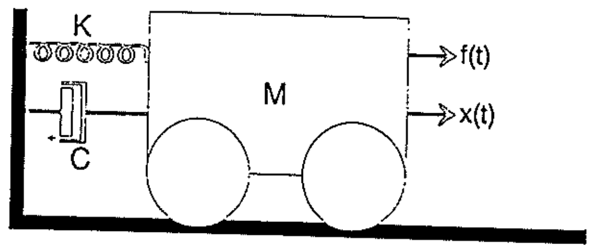

Fig.a.1.1: Single degree of freedom system

All present damping is approximated by a general viscous damping. Transforming this time domain equation into the Laplace domain (variable p), assuming the initial displacement and velocity are zero, yields:

(a.1.2) $$\left(Mp^2+Cp+K\right)X(p)=F(p)$$

or

(a.1.3) $$Z(p)\,X(p)=F(p)\,,$$

where Z is the dynamic stiffness

Inverting equation a.1.2 or a.1.3 results in the definition of the *transfer function*, $H(p)=Z^{-1}(p)$:

(a.1.4) $$X(p)=H(p)\,F(p),$$

(a.1.5) $$H(p)=\frac{1/M}{p^2+(C/M)p+(K/M)}.$$

This transfer function is a complex valued function (see figures a.1.2 to a.1.5).

A.1.1.2. System poles, natural frequencies, damping ratios

The denominator of equation a.1.5 is referred to as the *system characteristic equation*. Its roots, or the *system poles*, are:

(a.1.6) $$\lambda_{1,2}=-(C/(2M))\pm\sqrt{(C/(2M))^2-(K/M)}.$$

This equation enables the introduction of some important concepts. If there is no damping the system under consideration is a conservative system ($C=0$). From equation a.1.6, the *undamped natural frequency* (rad/s) is defined as:

(a.1.7) $$\Omega_1=\sqrt{K/M}.$$

The *critical damping*, C_c, is the damping value that makes the term under the square root in equation a.1.6 zero:

(a.1.8) $$C_c=2M\sqrt{K/M}.$$

Hence, *fraction of critical damping* or *damping ratio*, ζ_1, is:

(a.1.9) $$\zeta_1=C/C_c.$$

The damping is sometimes expressed as a quality- or Q-factor, which equals: $1/(2\zeta_1)$

Equation a.1.6 yields in the time domain an solution of the homogeneous system equation a.1.1:

(a.1.10) $$x(t)=x_1e^{\lambda_1 t}+x_2e^{\lambda_2 t}$$

Depending on the value of the damping ratio systems are classified as *overdamped* ($\zeta_1>1$), *critically damped* ($\zeta_1=1$) or *underdamped* ($\zeta_1<1$) systems. The response of

overdamped systems consist of a decay only. They have no tendency to oscillation. The response of underdamped systems is a decaying oscillation. Critically damped systems form the border case between over- and underdamped systems. For real-world systems the damping ratio is rarely larger than ten percent (0.1), unless the system contains some active damping mechanisms. Hence, only the underdamped case will be considered. In this case equation a.1.6 yields two complex conjugate roots:

(a.1.11) $$\lambda_1=\sigma_1+j\omega_1 \quad\text{and}\quad \lambda_1^*=\sigma_1-j\omega_1,$$

where σ_1 is the *damping factor* and ω_1 the *damped natural frequency*.

Other relations concerning the system poles are:

(a.1.12) $$\lambda_1=\left(-\zeta_1+j\sqrt{1-\zeta_1^2}\right)\Omega_1,$$

(a.1.13) $$\zeta_1=-\frac{\sigma_1}{\sqrt{\omega_1^2+\sigma_1^2}},$$

(a.1.14) $$\sigma_1=-\zeta_1\Omega_1,$$

(a.1.15) $$\Omega_1=\sqrt{\omega_1^2+\sigma_1^2}.$$

A.1.1.3. Residues

With the knowledge of equation a.1.11, equation a.1.5 for the transfer function becomes:

(a.1.16) $$H(p)=\frac{1/M}{(p-\lambda_1)(p-\lambda_1^*)}.$$

Applying the theory of the partial fraction expansion yields:

(a.1.17) $$H(p)=\frac{A_1}{(p-\lambda_1)}+\frac{A_1^*}{(p-\lambda_1^*)} \quad\text{with}\quad A_1=\frac{1/M}{j2\omega_1}.$$

In this formulation the terms A_1 and A_1^* are the *residues*.

A.1.1.4. Transfer function plots

Figures a.1.2 to a.1.5 display respectively the real part, the imaginary part, the magnitude and the phase of a typical single degree of freedom transfer function. Be aware of the different scaling of the axes !

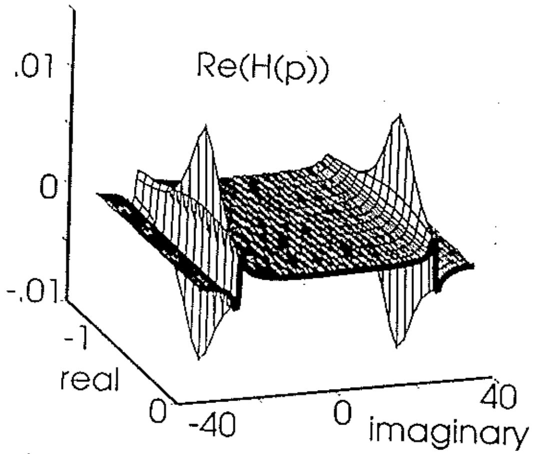

Fig.a.1.2: Real part of a single degree of freedom transfer function (axes in rad/s, transfer function value in mm/N)

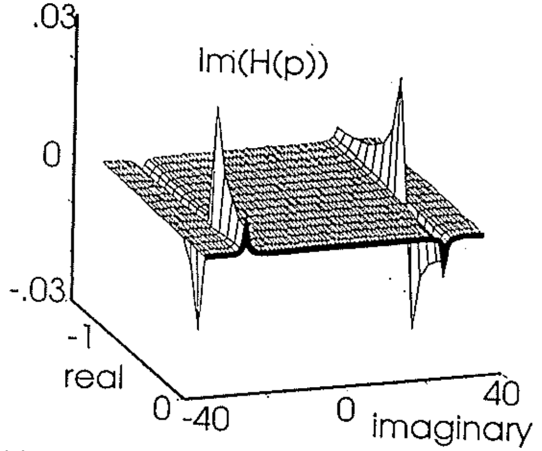

Fig.a.1.3: Imaginary part of a single degree of freedom transfer function (axes in rad/s, transfer function value in mm/N)

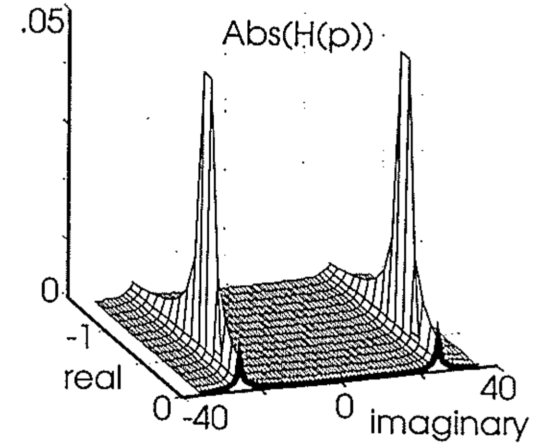

Fig.a.1.4: Magnitude of a single degree of freedom transfer function (axes in rad/s, transfer function value in mm/N)

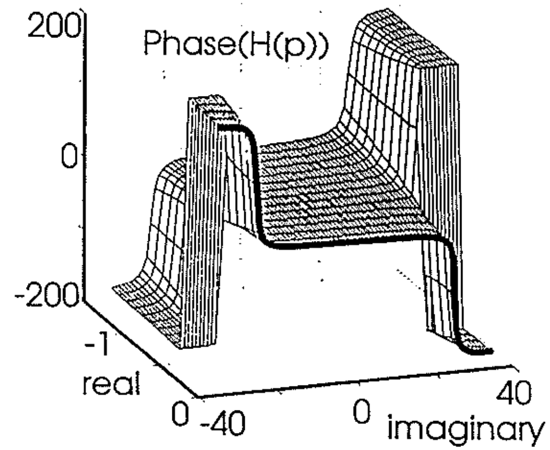

Fig.a.1.5: Phase of a single degree of freedom transfer function (axes in rad/s, transfer function phase in degrees)

A.1.1.5. Frequency response function, impulse response function

The previous sections discussed the relation between input (force) and output (displacement) of a single degree of freedom system in the Laplace domain. This relation can also be expressed in the frequency or time domain. The transfer function evaluated along the frequency axis ($j\omega$) is called the *frequency response function (FRF)*:

(a.1.18) $$H(p)\big|_{p=j\omega} = H(\omega) = \frac{A_1}{(j\omega-\lambda_1)} + \frac{A_1^*}{(j\omega-\lambda_1^*)}.$$

The FRF is a subset of the transfer function: it is the cross section along the frequency axis ($j\omega$, or σ=0). The contribution of the complex conjugate part (or negative frequency part) is negligible around resonance, $\omega \cong \omega_1$,. Therefor, the frequency response function for a single degree of freedom system is often approximated by:

(a.1.19) $$H(\omega) \cong \frac{A_1}{(j\omega-\lambda_1)}.$$

Section A.3.3 shows that these are the basic formulas for single degree of freedom parameter estimation methods. It shows how the modal parameters (system poles and residues) are estimated from measured frequency responses assuming single degree of freedom conditions.

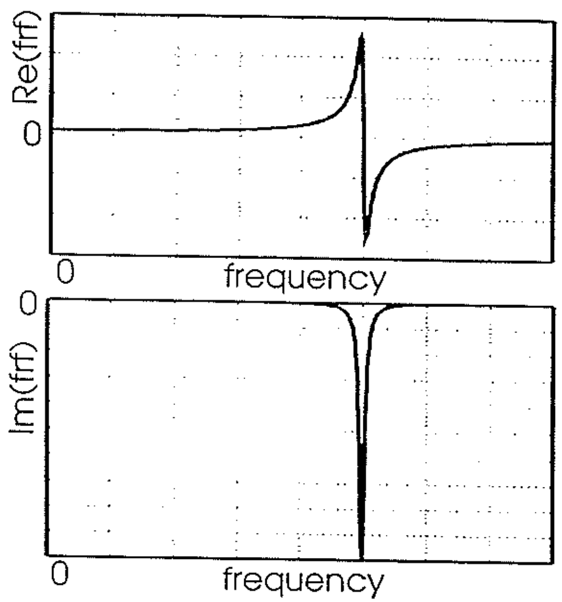

Fig.a.1.6: Real and imaginary part of a SDOF frequency response function

Figure a.1.6 shows the real and imaginary part of a typical frequency response function. These curves are equal to the positive frequency part of the thick full line on figures a.1.2 and a.1.3.

Inverse Laplace transforming the expression for the transfer function (eq.a.1.17) yields the expression in the time domain: the *impulse response function*:

(a.1.20) $$h(t) = A_1 e^{\lambda_1 t} + A_1^* e^{\lambda_1^* t} = e^{\sigma_1 t}\left(A_1 e^{j\omega_1 t} + A_1^* e^{-j\omega_1 t}\right).$$

The residue A_1 defines the initial amplitude, the real part of the pole, σ_1, the decay rate and the imaginary part of the pole, ω_1, the frequency of oscillation. The impulse response of a system is the system response to a Dirac impulse at time t=0. Figure a.1.7 displays a typical impulse response function.

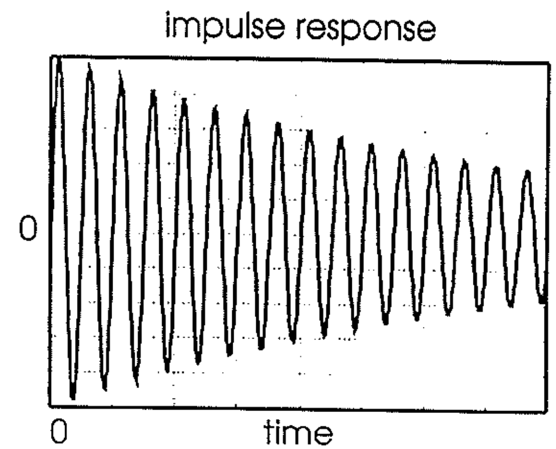

Fig.a.1.7: Typical SDOF impuls response function

A.1.1.6. Influence of mass, damping and stiffness changes

Figures a.1.8 to a.1.10 show how stiffness, damping and mass changes affect the frequency response of a single degree of freedom system.

A stiffness increase results in a higher resonance frequency and a lower FRF value in the low frequency range (fig.a.1.8). Because of this dominant stiffness influence at the low frequency end of the frequency response function this region is called the stiffness, or more accurately, the compliance line.

A damping increase causes a slight decrease in resonance frequency (ω_1). Its main influence however is a decrease in the amplitude of the frequency response function at resonance (fig.a.1.9). Also the phase changes more smoothly. If the damping is zero the magnitude at resonance becomes infinite and the phase suddenly drops 180 degrees. Furthermore, the system poles λ_1 become purely imaginary and in amplitude equal to the undamped natural frequency ($\Omega_1 = \sqrt{K/M}$).

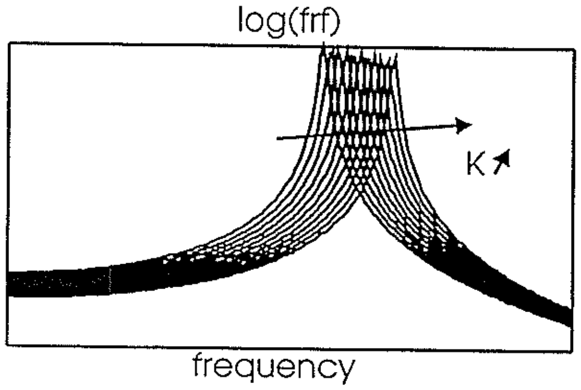

Fig.a.1.8: SDOF system: influence of a change of stiffness

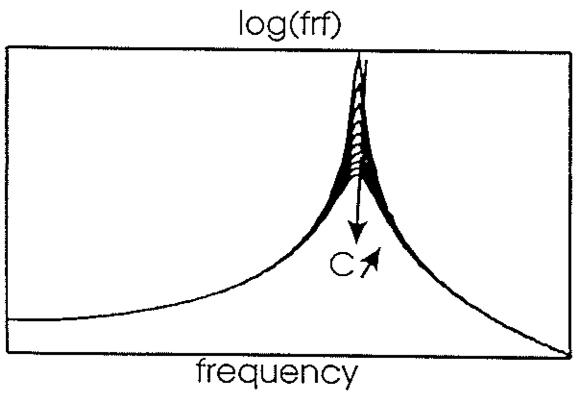

Fig.a.1.9: SDOF system: influence of a change of damping

An increasing mass shifts the resonance frequency to a lower value. The amplitude of the FRF at higher frequencies also decreases (fig.a.1.10). Due to this dominant effect at higher frequencies this area of a SDOF frequency response function is called the mass line.

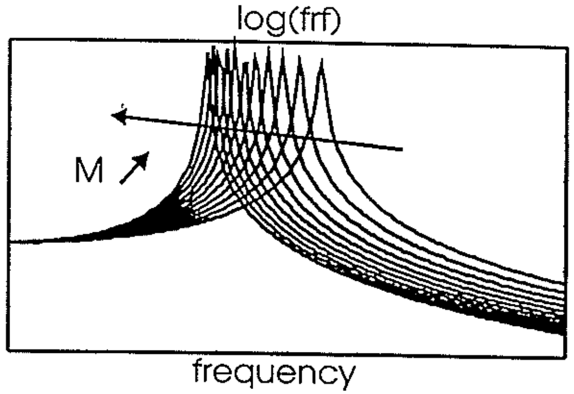

Fig.a.1.10: SDOF system: influence of a change of mass

A.1.2. MULTIPLE DEGREE OF FREEDOM SYSTEMS

This section will extend the concepts introduced for the single degree of freedom case. It will define the *natural frequencies* and *damping factors* from the *system poles*, defined by the denominator of the transfer function. Once the system poles are known, the transfer function can be expressed in partial fraction form, showing that the transfer function of a multiple degree of freedom system is a linear combination of transfer functions of single degree of freedom systems. Compared to the single degree of freedom system, in particular the concept of *residues* is extended. Sections A.1.2.3 and A.1.2.4 show the relation between residues, *mode shape vectors* and *modal participation factors*. The mode shape factors comply with *orthogonality* conditions with respect to the system matrices. These orthogonalities then define the *modal scale factors*. Section A.1.2.6 discusses some special damping cases: systems without damping and systems with proportional damping.

A.1.2.1. System equation, transfer function

In most cases the system under study can not be described as a single degree of freedom system. Most systems consist of a (continuous) assembly of an infinite number of masses, stifnesses and damping. The following paragraphs will show how the transfer functions of such a multiple degree of freedom (MDOF) system are related to its modal parameters (resonance frequencies and modal vectors).

In order to show that for MDOF systems the simple algebraic equation, expressing the force equilibrium of a SDOF system (eq.a.1.1) evolves into a matrix equation of a similar form, the two degree of freedom system of figure a.1.11 serves as an example.

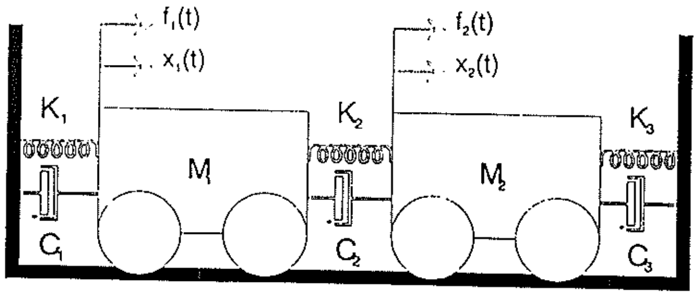

Fig.a.1.11: Example two degree of freedom system

The equations of motion of this system are:

(a.1.21)
$$M_1\ddot{x}_1(t)+(C_1+C_2)\dot{x}_1(t)-C_2\dot{x}_2(t)+(K_1+K_2)x_1(t)-K_2x_2(t)=f_1(t)$$
$$M_2\ddot{x}_2(t)+(C_2+C_3)\dot{x}_2(t)-C_2\dot{x}_1(t)+(K_2+K_3)x_2(t)-K_2x_1(t)=f_2(t)$$

In matrix notation this becomes (deleting (t) for readability):

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix}+\begin{bmatrix} C_1+C_2 & -C_2 \\ -C_2 & C_2+C_3 \end{bmatrix}\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix}+\begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2+K_3 \end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}f_1\\ f_2\end{Bmatrix}$$
(a.1.22)

or:

(a.1.23) $$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f\},$$

where: $[M]$: the mass matrix
$[C]$: the damping matrix
$[K]$: the stiffness matrix
$\{f(t)\}$: the forcing vector
$\{x(t)\}$: the response vector.

This equation also describes the behaviour of systems with more degrees of freedom. The matrix dimensions increase accordingly. Transforming this time domain matrix equation into the Laplace domain (variable p), assuming the initial displacements and velocities are zero, yields:

(a.1.24) $$\left(p^2[M]+p[C]+[K]\right)\{X(p)\}=\{F(p)\}$$

or

(a.1.25) $$[Z(p)]\{X(p)\}=\{F(p)\},$$

where $[Z(p)]$ is the dynamic stiffness matrix.

Inverting equation a.1.24 or a.1.25 results in the definition of the *transfer function matrix*, $[H(p)]$:

(a.1.26) $$\{X(p)\}=[H(p)]\{F(p)\},$$

Standard calculus proves that the inverse of a matrix can be calculated from its adjoint matrix:

(a.1.27) $$[H(p)]=[Z(p)]^{-1}=\frac{adj([Z(p)])}{|Z(p)|},$$

where: $adj([Z(p)])$: the adjoint matrix of $[Z(p)]$, $=\left[\varepsilon_{ij}\left|Z_{ij}\right|\right]'$,
$\left|Z_{ij}\right|$: the determinant of $[Z(p)]$, without row i and column j,
ε_{ij}: $=1$, if $i+j$ is even; $=-1$, if $i+j$ is odd,
$|Z(p)|$: the determinant of $[Z(p)]$.

The transfer function matrix contains complex valued functions.

A.1.2.2. System poles, natural frequencies, damping factors

The denominator of equation a.1.27, the determinant of $[Z(p)]$, is the *system characteristic equation*. As is the case for single degree of freedom systems the roots of this system characteristic equation, or *system poles*, define the resonance frequencies of the system. These roots can be found based on an eigenvalue problem. In order to transform the system equation a.1.24 into a general eigenvalue problem formulation, the following identity is added:

(a.1.28) $$\left(p[M]-p[M]\right)\{X\}=\{0\}.$$

The combination of equations a.1.24 and a.1.28 results in:

(a.1.29) $$\left(p[A]+[B]\right)\{Y\}=\{F'\},$$

where:

$$[A]=\begin{bmatrix}[0] & [M]\\ [M] & [C]\end{bmatrix},\ [B]=\begin{bmatrix}-[M] & [0]\\ [0] & [K]\end{bmatrix},\ \{Y\}=\begin{Bmatrix}p\{X\}\\ \{X\}\end{Bmatrix}\text{ and }\{F'\}=\begin{Bmatrix}\{0\}\\ \{F\}\end{Bmatrix}.$$

If the forcing function equals zero, equation a.1.29 formulates a general eigenvalue problem with real valued matrices. Its eigenvalues are the values of p complying with following equation:

(a.1.30) $$\left|p[A]+[B]\right|=0.$$

Elaborating this equation shows that the roots of this equation are the roots of the above mentioned characteristic equation ($|Z(p)|=0$). This equation generates $2N$ (N = number of degrees of freedom) complex valued eigenvalues, appearing in complex conjugate pairs:

(a.1.31) $$[{}^{\backslash}\Lambda_{\backslash}]=\begin{bmatrix}\lambda_1 & & & & & 0\\ & \ddots & & & & \\ & & \lambda_N & & & \\ & & & \lambda_1^* & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_N^*\end{bmatrix}=\begin{bmatrix}\sigma_1+j\omega_1 & & & & & 0\\ & \ddots & & & & \\ & & \sigma_N+j\omega_N & & & \\ & & & \sigma_1-j\omega_1 & & \\ & & & & \ddots & \\ 0 & & & & & \sigma_N-j\omega_N\end{bmatrix}$$

As is the case for the single degree of freedom system, the real part of a pole, σ_r, is the *damping factor*, the imaginary part, ω_r, the *damped natural frequency*.

A.1.2.3. Modal vectors, residues

With the above mentioned eigenvalues corresponds a set of eigenvectors. For multiple degree of freedom systems these eigenvectors introduce the concept of *mode shape vectors, modal displacement vectors* or *modal vectors*, $\{\psi\}_r$. They also appear in complex conjugate pairs. Each eigenvector corresponds with a specific eigenvalue.

(a.1.32) $$[\Phi]=\begin{bmatrix}\lambda_1\{\psi\}_1 & \cdots & \lambda_N\{\psi\}_N & \lambda_1^*\{\psi\}_1^* & \cdots & \lambda_N^*\{\psi\}_N^*\\ \{\psi\}_1 & \cdots & \{\psi\}_N & \{\psi\}_1^* & \cdots & \{\psi\}_N^*\end{bmatrix}.$$

In general these modal vectors contain complex valued modal displacements. The phases of these vector elements can differ. At the corresponding pole, λ_r, these vectors (since they are eigenvectors) make the forcing vector, $\{F\}$, in the system equation a.1.24 equal to $\{0\}$:

(a.1.33) $$\left(\lambda_r^2[M]+\lambda_r[C]+[K]\right)\{\psi\}_r=[Z(\lambda_r)]\{\psi\}_r=\{0\}.$$

Similarly as for the single degree of freedom system the concept of residues is introduced. Since λ_r, λ_r^* $(r=1,N)$ are the roots of the system characteristic equation $|Z(p)|$, equation a.1.27 for the transfer function can be rewritten as:

(a.1.34) $$[H(p)]=\frac{adj([Z(p)])}{\prod_{r=1}^{N}E(p-\lambda_r)(p-\lambda_r^*)}=\frac{adj([Z(p)])}{\prod_{r=1}^{2N}E(p-\lambda_r)},$$

where: E: a constant,
$\lambda_{N+s}=\lambda_s^*$ for $s=1,N$.

Applying the theory of partial fraction expansion yields:

(a.1.35) $$[H(p)]=\sum_{r=1}^{N}\left(\frac{[A]_r}{(p-\lambda_r)}+\frac{[A]_r^*}{(p-\lambda_r^*)}\right)$$

The terms $[A]_r$ and $[A]_r^*$ are called the *residues*. From calculus it is known that these residues are equal to:

(a.1.36) $$[A]_r=\left([H(p)](p-\lambda_r)\right)\Big|_{p=\lambda_r}.$$

The following derivation will show the relation between residues and mode shape vectors. Introducing equation a.1.34 into equation a.1.36:

(a.1.37) $$[A]_r=\frac{adj([Z(\lambda_r)])}{\prod_{s=1,s\neq r}^{2N}E(\lambda_r-\lambda_s)}\quad\text{or}$$

$$[A]_r=P_r.adj([Z(\lambda_r)]),$$

where P_r is a pole dependent constant.

Hence equation a.1.35 becomes:

(a.1.38) $$[H(p)]=\sum_{r=1}^{N}\left(\frac{P_r.adj[Z(\lambda_r)]}{(p-\lambda_r)}+\frac{P_r^*.adj[Z(\lambda_r)]^*}{(p-\lambda_r^*)}\right).$$

A closer look at the term $adj([Z(\lambda_r)])$ will clarify the relation between the transfer function matrix, $[H(p)]$, and the modal vectors, $\{\psi\}_r$. Rewriting expression a.1.27 for the transfer function matrix yields:

(a.1.39) $$[Z(p)].adj([Z(p)])=|Z(p)|[{}^{\backslash}I_{\backslash}].$$

Evaluating this equation for $p=\lambda_r$ gives, since λ_r is a root of the characteristic equation:

(a.1.40) $$[Z(\lambda_r)]\, adj([Z(\lambda_r)]) = [0].$$

Considering any arbitrary column (e.g. column i) of $adj([Z(\lambda_r)])$ yields:

(a.1.41) $$[Z(\lambda_r)]\left\{adj([Z(\lambda_r)])\right\}_i = \{0\}.$$

This equation is equal to equation a.1.33, expressing the homogeneous equation for the eigenvector $\{\psi\}_r$. Hence, $\left\{adj([Z(\lambda_r)])\right\}_i$ and $\{\psi\}_r$ are proportional and both represent the eigenvector corresponding to the eigenvalue λ_r. This is true for any arbitrary column of $adj([Z(\lambda_r)])$. Hence, since all columns of $adj([Z(\lambda_r)])$ are proportional to each other, $adj([Z(\lambda_r)])$ is of rank 1. This also means that all rows of $adj([Z(\lambda_r)])$ are proportional to each other. Therefore, the adjoint matrix, evaluated at λ_r, meets the following condition:

(a.1.42) $$adj([Z(\lambda_r)]) \div \{\psi\}_r \langle L\rangle_r.$$

Since mass, stiffness and damping matrix are symmetric for systems that comply with Maxwell's reciprocity, the dynamic stiffness matrix, $[Z(p)]$, is symmetric and its adjoint matrix too. Hence, also the rows of $adj([Z(\lambda_r)])$ are proportional to the r-th modal vector:

(a.1.43) $$adj([Z(\lambda_r)]) = R_r\{\psi\}_r\{\psi\}_r^t = R_r\begin{bmatrix}\psi_1\psi_1 & \psi_1\psi_2 & \cdots & \psi_1\psi_N\\ \psi_2\psi_1 & \psi_2\psi_2 & \cdots & \psi_2\psi_N\\ \vdots & \vdots & \ddots & \vdots\\ \psi_N\psi_1 & \psi_N\psi_2 & \cdots & \psi_N\psi_N\end{bmatrix}_r,$$

where: R_r: a constant associated with the scaling of $\{\psi\}_r$,
superscript ^{t}: indicates the transpose.

Fitting this expression into equation a.1.38, combining the constants R_r and P_r into $Q_r = P_r.R_r$, yields:

(a.1.44) $$[H(p)] = \sum_{r=1}^{N}\left(\frac{Q_r\{\psi\}_r\{\psi\}_r^t}{(p-\lambda_r)} + \frac{Q_r^*\{\psi\}_r^*\{\psi\}_r^{*t}}{(p-\lambda_r^*)}\right).$$

Hence, each residue

(a.1.45) $$[A]_r = Q_r\{\psi\}_r\{\psi\}_r^t.$$

Since all columns of $[A]_r$ are proportional to each other, each column j contains sufficient information to construct this matrix, except if this column happens to coincide with a modal coefficient ψ_{jr} equal to zero. The corresponding row and column will than be zero. In practice this means that a mode can not be detected if the excitation is at a nodal point of that mode. A proper selection of the excitation degrees of freedom will avoid this problem (sections A.6.2.4 and B.2.2.2).

Equation a.1.35 shows that the residues $[A]_r$ are absolute quantities. Therefore, equation a.1.45 illustrates that the modal vectors $\{\psi\}_r$ are scaled vectors and depend on the scaling factors Q_r.

A.1.2.4. Modal participation factors

With $[V] = \left[\{\psi\}_1 \ \cdots \ \{\psi\}_N \ \ \{\psi\}_1^* \ \cdots \ \{\psi\}_N^*\right]$,

$\left[p[{}^{\backslash}I_{\backslash}] - [{}^{\backslash}\Lambda_{\backslash}]\right]^{-1}$ containing the terms $\dfrac{1}{(p-\lambda_r)}$ and $\dfrac{1}{(p-\lambda_r^*)}$ and

$[L] = \left[Q_1\{\psi\}_1 \ \cdots \ Q_N\{\psi\}_N \ \ Q_1^*\{\psi\}_1^* \ \cdots \ Q_N^*\{\psi\}_N^*\right]^t = [{}^{\backslash}Q_{\backslash}][V]^t$

equation a.1.44 becomes:

(a.1.46) $$[H(p)] = [V]\left[p[{}^{\backslash}I_{\backslash}] - [{}^{\backslash}\Lambda_{\backslash}]\right]^{-1}.[L] \text{ or}$$
$$[H(p)] = [V]\left[p[{}^{\backslash}I_{\backslash}] - [{}^{\backslash}\Lambda_{\backslash}]\right]^{-1}.[{}^{\backslash}Q_{\backslash}][V]^t.$$

Expressing the relation between input forces $\{F(p)\}$ and output displacements $\{X(p)\}$ based upon this equation, yields:

(a.1.47) $$\{X(p)\} = [H(p)]\{F(p)\} = [V]\left[p[{}^{\backslash}I_{\backslash}] - [{}^{\backslash}\Lambda_{\backslash}]\right]^{-1}.[L]\{F(p)\}.$$

In this context the matrix $[V]$ relates to the responses or displacements. This matrix is the *modal vector matrix*. The matrix $[L]$ relates to the inputs or the forces and is the *modal participation factor matrix*. It contains the transposes of the modal vectors multiplied with the corresponding scaling factors Q. As such it is a measure of the efficiency of excitation of each ecitation degree of freedom for each mode (sections A.4.2 and B.4.2.3).

A.1.2.5. Frequency response function matrix, impulse response function matrix

The transfer function matrix evaluated along the frequency axis ($j\omega$) is the *frequency response function matrix*. Hence, equation a.1.44 yields:

(a.1.48) $$[H(j\omega)] = \sum_{r=1}^{N}\left(\frac{Q_r\{\psi\}_r\{\psi\}_r^t}{(j\omega-\lambda_r)} + \frac{Q_r^*\{\psi\}_r^*\{\psi\}_r^{*t}}{(j\omega-\lambda_r^*)}\right).$$

or, according to equation a.1.47:

(a.1.49) $$[H(j\omega)] = [V]\left[j\omega[{}^{\backslash}I_{\backslash}] - [{}^{\backslash}\Lambda_{\backslash}]\right]^{-1}.[L] \text{ and}$$
$$\{X(j\omega)\} = [V]\left[j\omega[{}^{\backslash}I_{\backslash}] - [{}^{\backslash}\Lambda_{\backslash}]\right]^{-1}.[L]\{F(j\omega)\}$$

Experimental modal analysis will never measure the complete frequency response function matrix $[H(j\omega)]$, due to practical limitations. The number of modes, N_m, will always be smaller than the number of response points, N_o. This number of response points is also smaller than the number of degrees of freedom, N. In most cases the number of inputs, N_i, is a lot smaller than the number of responses, typically between 1 and 5. Therefore, these practical approximations will yield matrices in equation a.1.49 with the following dimensions:

(a.1.50) $$\{X(j\omega)\}_{N_o\times 1} = [H(j\omega)]_{N_o\times N_i}\{F(j\omega)\}_{N_i\times 1} \text{ or}$$
$$\{X(j\omega)\}_{N_o\times 1} = [V]_{N_o\times 2N_m}.\left[j\omega[{}^{\backslash}I_{\backslash}] - [{}^{\backslash}\Lambda_{\backslash}]\right]^{-1}_{2N_m\times 2N_m}.[L]_{2N_m\times N_i}\{F(j\omega)\}_{N_i\times 1}.$$

According to equation a.1.48 the frequency response function can be interpreted as the sum of a number of components, each equivalent to the response of a single degree of freedom system (fig.a.1.12 (ref.a.1.3)).

The frequency response function can be represented in several ways: real and imaginary part as function of the frequency (fig.a.1.13a), amplitude (often on a logarithmic scale) and phase (Bode plot, fig.a.1.13b), and real part versus imaginary part, with the frequency as varying parameter over the curve (Nyquist or Argand plot, fig.a.1.13c).

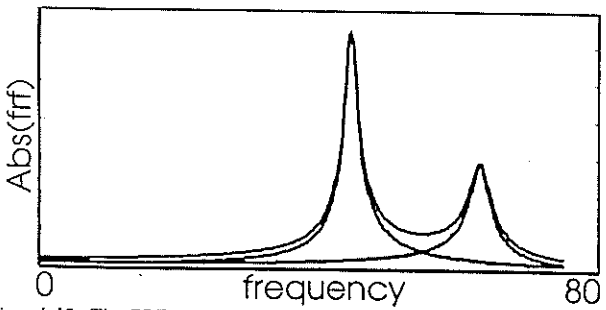

Fig.a.1.12: The FRF as a sum of single degree of freedom system FRF's.

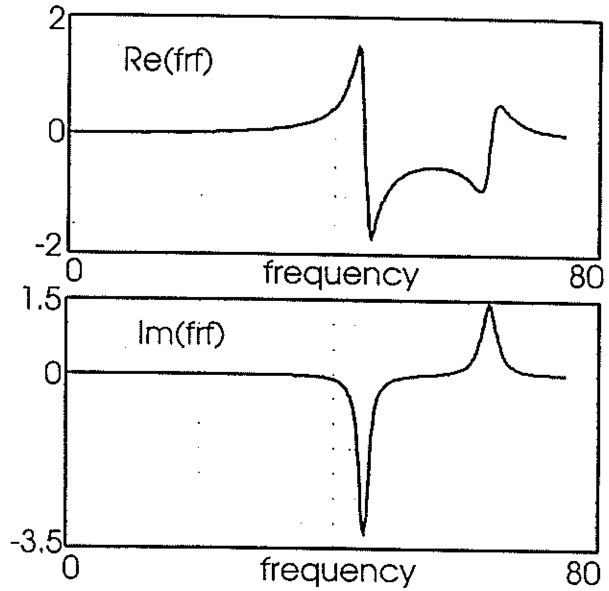

Fig.a.1.13a: Real and imaginary part of a FRF.

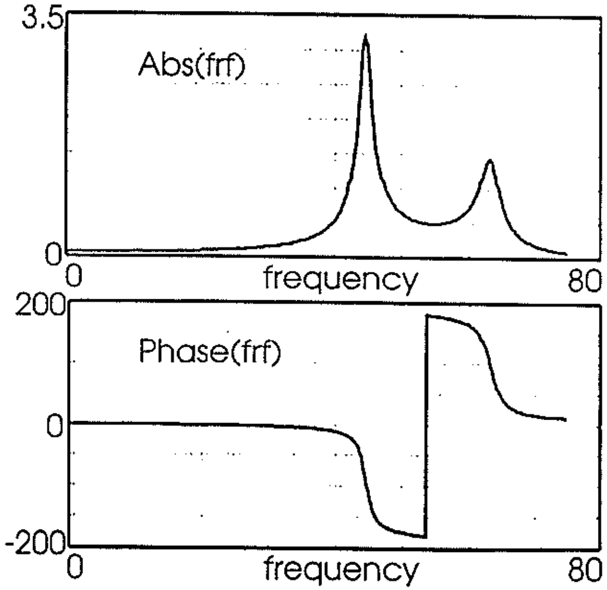

Fig.a.1.13b: Amplitude and phase of a FRF.

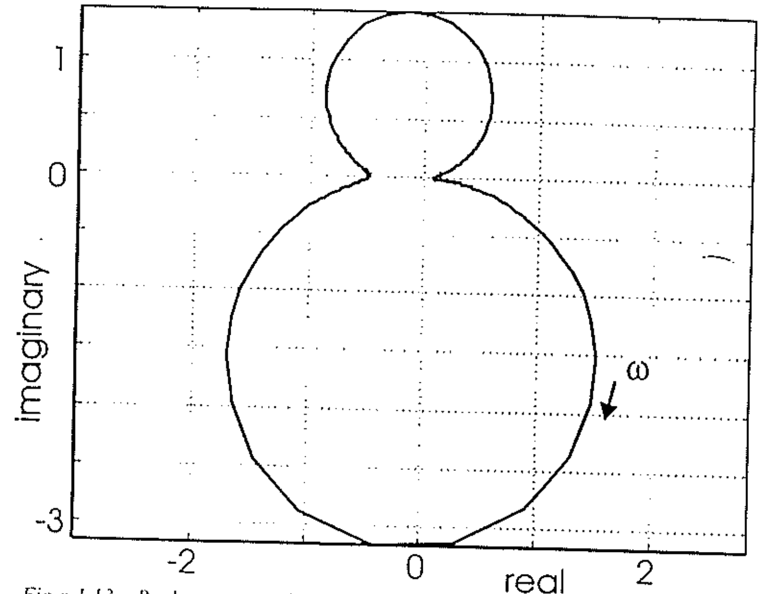

Fig.a.1.13c: Real part versus imaginary part of a FRF (Nyquist or Argand plot).

Inverse Laplace transforming the expression for the transfer function matrix yields the expression in the time domain: the *impulse response function matrix*:

(a.1.51) $$[h(t)] = \sum_{r=1}^{N} \left(Q_r \{\psi\}_r \{\psi\}_r^t e^{\lambda_r t} + Q_r^* \{\psi\}_r^* \{\psi\}_r^{*t} e^{\lambda_r^* t} \right),$$

or, according to equation a.1.47:

(a.1.52) $$[H(t)] = [V]\left[{}^{\backslash} e^{\Lambda t}{}_{\backslash}\right][L],$$

$$\text{where } \left[{}^{\backslash} e^{\Lambda t}{}_{\backslash}\right] = \begin{bmatrix} e^{\lambda_1 t} & & & & 0 \\ & \ddots & & & \\ & & e^{\lambda_N t} & & \\ & & & e^{\lambda_1^* t} & \\ 0 & & & & \ddots \\ & & & & & e^{\lambda_N^* t} \end{bmatrix}.$$

A.1.2.6. Undamped and proportionally damped systems

The previous sections discussed the quite general case, where all present damping is approximated as a general viscous damping. These damped systems yield complex

valued resonance frequencies (system poles), complex valued modal vectors, with different phase for each vector element, and complex valued frequency responses. The following sections will discuss some special damping cases: no damping and proportional damping.

If the damping matrix, [C], is zero, the system equation in the Laplace domain (eq.a.1.24) becomes:

(a.1.53) $$\left(p^2[M]+[K]\right)\{X\} = \{F\}$$

The procedures, explained for the general viscous damping case, will yield *purely imaginary* poles (the damping factors, σ_r, being zero) in complex conjugate pairs :

(a.1.54) $$\lambda_1 = j\omega_1, \cdots, \lambda_N = j\omega_N, \lambda_1^* = -j\omega_1, \cdots, \lambda_N^* = -j\omega_N .$$

In general the corresponding mode shape vectors, $\{\psi\}_r$, will contain complex modal displacements, but the phases of the elements of a specific modal vector will be equal or differ exactly 180°. Since the mode shape vectors depend on the choice of the scale factors Q_r (eq.a.1.45), these modal vectors can be scaled into purely real valued vectors. Therefore, in the undamped case the mode shape vectors are called real or *normal modal vectors*. The modal model validation techniques 'modal phase collinearity' and 'mean phase deviation' are based on this characteristic (sections A.4.5 and B.4.2.4).

In the no-damping case the roots of the characteristic equation, $\left|p^2[M]+[K]\right|$, are more easily found in the solution of the eigenvalue problem:

(a.1.55) $$\left(p^2[M]+[K]\right)\{X\} = \{0\}.$$

This solution procedure directly yields the eigenvalues, $-\omega_r^2$, and the correspondig normal modal vectors, $\{\psi\}_r$. The above mentioned roots of the characteristic equation are the square roots of the eigenvalues.

The dynamic stiffness matrix in the frequency domain ($p = j\omega$),

$$[Z(j\omega)] = -\omega^2[M]+[K],$$

is real valued. Hence, its inverse, the frequency response function matrix $[H(j\omega)]$, is also a real valued matrix. The general expression a.1.48 can be simplified into:

(a.1.56) $$[H(j\omega)] = \sum_{r=1}^{N} \frac{j2\omega_r Q_r \{\psi\}_r \{\psi\}_r^t}{\left(\omega_r^2 - \omega^2\right)} \quad \text{or}$$

$$[H(j\omega)] = [\psi]\left[\left[{}^{\backslash}\omega_r^2{}_{\backslash}\right] - \omega^2\left[{}^{\backslash}I_{\backslash}\right]\right]^{-1}\left[{}^{\backslash} j2\omega_r Q_r{}_{\backslash}\right][\psi]^t$$

Note that, since the frequency response function is real valued, the quantity $Q_r\{\psi\}_r\{\psi\}_r^t$ is purely imaginary.

The proportional damping case is a quite hypothetical form of damping. In its most simple form the damping matrix, $[C]$, complies with:

(a.1.57) $$[C]=\alpha[M]+\beta[K].$$

where α and β are real constants.

The system equation changes into:

(a.1.58) $$\left(p^2[M]+p(\alpha[M]+\beta[K])+[K]\right)\{X\}=\{F\},$$
$$\left((p^2+p\alpha)[M]+(p\beta+1)[K]\right)\{X\}=\{F\}.$$

Considering the homogeneous version of this equation and dividing it by $(p\beta+1)$ yields:

(a.1.59) $$\left(\left(\frac{p^2+p\alpha}{p\beta+1}\right)[M]+[K]\right)\{X\}=\{0\}.$$

This equation is similar to equation a.1.55 for the undamped system. Hence, systems with proportional damping have *complex* system poles, λ_r, complying with

(a.1.60) $$\frac{\lambda_r^2+\lambda_r\alpha}{\lambda_r\beta+1}=-\omega_r^2,$$

and *normal* modal vectors, equal to these of the undamped system. Hence, the numerical complexity of the calculations with proportionally damped systems is lower than with the general viscous damping case. This is the main reason for the introduction of proportionally damped systems. They form a compromise between the undamped system models from a finite element model analysis and the generally viscously damped system models from experimental modal analysis. The frequency response functions are complex functions. Similarly as for the undamped system, the general expression a.1.48 can be simplified into:

(a.1.61) $$[H(j\omega)]=\sum_{r=1}^{N}\frac{j2\omega_r Q_r\{\psi\}_r\{\psi\}_r^t}{(\sigma_r^2+\omega_r^2-\omega^2)-2\sigma_r j\omega}\quad\text{or}$$
$$[H(j\omega)]=[\psi]\left[\left[{}^\backslash\sigma_r^2+\omega_r^2{}_\backslash\right]-\omega^2\left[{}^\backslash I_\backslash\right]-2j\omega\left[{}^\backslash\sigma_{r\backslash}\right]\right]^{-1}\left[{}^\backslash j2\omega_r Q_\backslash\right][\psi]^t$$

Systems with a damping matrix complying with:

(a.1.62) $$\left[[M]^{-1}[C]\right]^t\left[[M]^{-1}[K]\right]^t=\left[[M]^{-1}[K]\right]^t\left[[M]^{-1}[C]\right]^t,$$

have the same properties.

A.1.2.7. Orthogonality, modal coordinates.

This section will discuss the orthogonality properties of the modal vectors. It will show that these properties form the base for a transformation that will decouple the system equations of motion. This transformation enables another approach to derive the frequency response matrix in terms of modal parameters (system poles and modal vectors). For the general viscous damping case this discussion yields the rather abstract concept of *modal a* and *modal b*. The discussion of the proportionally or undamped case will define the *modal mass*, *stiffness* and *damping*, which can be interpreted as the mass stiffness and damping of the single degree of freedom systems, defined in the modal coordinate vector space.

General viscous damping case:

The discussion about orthogonality of modal vectors for the general viscous damping case starts from equations a.1.29 and a.1.32:

(a.1.29) $$(p[A]+[B])\{Y\}=\{F'\},$$

(a.1.32) $$[\Phi]=\begin{bmatrix}\lambda_1\{\psi\}_1 & \cdots & \lambda_N\{\psi\}_N & \lambda_1^*\{\psi\}_1^* & \cdots & \lambda_N^*\{\psi\}_N^* \\ \{\psi\}_1 & \cdots & \{\psi\}_N & \{\psi\}_1^* & \cdots & \{\psi\}_N^*\end{bmatrix}.$$

Substituting $p=\lambda_r$ and $\{Y\}=\begin{Bmatrix}\lambda_r\{\psi\}_r\\ \{\psi\}_r\end{Bmatrix}$ into equation a.1.29 yields (since these form an eigenvalue pair):

(a.1.63) $$(\lambda_r[A]+[B])\begin{Bmatrix}\lambda_r\{\psi\}_r\\ \{\psi\}_r\end{Bmatrix}=\{0\}.$$

Premultiplying with the transpose of another column, s, of $[\Phi]$ results in:

(a.1.64) $$\left\{\lambda_s\{\psi\}_s^t\quad\{\psi\}_s^t\right\}(\lambda_r[A]+[B])\begin{Bmatrix}\lambda_r\{\psi\}_r\\ \{\psi\}_r\end{Bmatrix}=\{0\}.$$

Reversing the role of the indices r and s, yields:

(a.1.65) $$\{\lambda_r\{\psi\}_r^t \quad \{\psi\}_r^t\}(\lambda_s[A]+[B])\begin{Bmatrix}\lambda_s\{\psi\}_s\\ \{\psi\}_s\end{Bmatrix}=\{0\}.$$

The transpose of this equation is:

(a.1.66) $$\{\lambda_s\{\psi\}_s^t \quad \{\psi\}_s^t\}(\lambda_r[A]+[B])\begin{Bmatrix}\lambda_r\{\psi\}_r\\ \{\psi\}_r\end{Bmatrix}=\{0\}.$$

Substracting this equation from equation a.1.64 results in:

(a.1.67) $$\{\lambda_s\{\psi\}_s^t \quad \{\psi\}_s^t\}((\lambda_r-\lambda_s)[A])\begin{Bmatrix}\lambda_r\{\psi\}_r\\ \{\psi\}_r\end{Bmatrix}=\{0\}.$$

If $\lambda_r \neq \lambda_s$, this equation shows the orthogonality of two different columns, r and s, of $[\Phi]$, if weighed by the matrix $[A]$. Introducing this orthogonality into equation a.1.66 proves that these vectors are also orthogonal if weighed by the matrix $[B]$. This yields following orthogonality conditions (see also the definitions in equation a.1.29):

(a.1.68) $$\begin{bmatrix}\lambda_1\{\psi\}_1^t & \{\psi\}_1^t\\ \vdots & \vdots\\ \lambda_N^*\{\psi\}_N^{*t} & \{\psi\}_N^{*t}\end{bmatrix}\begin{bmatrix}[0] & [M]\\ [M] & [C]\end{bmatrix}\begin{bmatrix}\lambda_1\{\psi\}_1 & \cdots & \lambda_N^*\{\psi\}_N^*\\ \{\psi\}_1 & \cdots & \{\psi\}_N^*\end{bmatrix}=[{}^{\backslash}a_{\backslash}]$$

and

(a.1.69) $$\begin{bmatrix}\lambda_1\{\psi\}_1^t & \{\psi\}_1^t\\ \vdots & \vdots\\ \lambda_N^*\{\psi\}_N^{*t} & \{\psi\}_N^{*t}\end{bmatrix}\begin{bmatrix}-[M] & [0]\\ [0] & [K]\end{bmatrix}\begin{bmatrix}\lambda_1\{\psi\}_1 & \cdots & \lambda_N^*\{\psi\}_N^*\\ \{\psi\}_1 & \cdots & \{\psi\}_N^*\end{bmatrix}=[{}^{\backslash}b_{\backslash}].$$

Evaluation of these orthogonality conditions generates following expressions:

(a.1.70) $$[V]^t[M][V][{}^{\backslash}\Lambda_{\backslash}]+[{}^{\backslash}\Lambda_{\backslash}][V]^t[M][V]+[V]^t[C][V]=[{}^{\backslash}a_{\backslash}]$$

and

(a.1.71) $$-[{}^{\backslash}\Lambda_{\backslash}][V]^t[M][V][{}^{\backslash}\Lambda_{\backslash}]+[V]^t[K][V]=[{}^{\backslash}b_{\backslash}].$$

where $[V]=[\{\psi\}_1 \ \cdots \ \{\psi\}_N \ \{\psi\}_1^* \ \cdots \ \{\psi\}_N^*]$,

$$[{}^{\backslash}\Lambda_{\backslash}]=\begin{bmatrix}\lambda_1 & & & & & 0\\ & \ddots & & & & \\ & & \lambda_N & & & \\ & & & \lambda_1^* & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_N^*\end{bmatrix}.$$

The diagonal matrices $[{}^{\backslash}a_{\backslash}]$ and $[{}^{\backslash}b_{\backslash}]$ are respectively the *modal a matrix* and the *modal b matrix*. The *modal* a_r and *modal* b_r depend on the modal scaling of mode r. These orthogonality conditions are the base for the transformation to *modal coordinates* $\{q\}$.

Introducing the transformation $\{Y\}=[\Phi]\{q\}$ into the system equation a.1.29 and premultiplication with $[\Phi]^t$, yields:

(a.1.72) $$(p[\Phi]^t[A][\Phi]+[\Phi]^t[B][\Phi])\{q\}=[\Phi]^t\{F'\}.$$

A very important feature of the modal vectors is that the above mentioned orthogonality properties make this a set of decoupled equations:

(a.1.73) $$(p[{}^{\backslash}a_{\backslash}]+[{}^{\backslash}b_{\backslash}])\{q\}=[\Phi]^t\{F'\}.$$

Evaluating these equations at all poles proves that:

(a.1.74) $$[{}^{\backslash}b_{\backslash}]=-[{}^{\backslash}a_{\backslash}][{}^{\backslash}\Lambda_{\backslash}].$$

Solving equation a.1.73 for $\{q\}$ and premultiplying the result with $[\Phi]$ yields:

(a.1.75) $$\begin{Bmatrix}p\{X\}\\ \{X\}\end{Bmatrix}=[\Phi]\{q\}=\begin{bmatrix}[V][{}^{\backslash}\Lambda_{\backslash}]\\ [V]\end{bmatrix}[p[{}^{\backslash}a_{\backslash}]-[{}^{\backslash}a_{\backslash}][{}^{\backslash}\Lambda_{\backslash}]]^{-1}\begin{bmatrix}[V][{}^{\backslash}\Lambda_{\backslash}]\\ [V]\end{bmatrix}^t\begin{Bmatrix}\{0\}\\ \{F\}\end{Bmatrix}.$$

The lower half of this equation, relating $\{X\}$ to $\{F\}$, gives an equation very similar to the matrix expression for the transfer function matrix, a.1.46:

(a.1.76) $$[H]=[V][p[{}^{\backslash}I_{\backslash}]-[{}^{\backslash}\Lambda_{\backslash}]]^{-1}[{}^{\backslash}a_{\backslash}]^{-1}[V]^t \text{ or}$$
$$[H]=[V][p[{}^{\backslash}I_{\backslash}]-[{}^{\backslash}\Lambda_{\backslash}]]^{-1}[L].$$

The undamped and proportionally damped case:

Starting from the simplified eigenvalue problem (eq.a.1.55) related to the undamped system equations a.1.53, a similar reasoning yields the definition of the *modal mass matrix*, $[{}^{\backslash}m_{\backslash}]$, and the *modal stiffness matrix*, $[{}^{\backslash}k_{\backslash}]$

(with $[\psi]=[\{\psi\}_1 \quad \cdots \quad \{\psi\}_N]$):

(a.1.77) $$[\psi]^t[M][\psi]=[{}^{\backslash}m_{\backslash}].$$

(a.1.78) $$[\psi]^t[K][\psi]=[{}^{\backslash}k_{\backslash}].$$

The modal coordinate transformation, $\{X\}=[\psi]\{q\}$, results in following set of decoupled equations:

(a.1.79) $$\left(-\omega^2[{}^{\backslash}m_{\backslash}]+[{}^{\backslash}k_{\backslash}]\right)\{q\}=[\psi]^t\{F\},$$

where $[{}^{\backslash}k_{\backslash}]=[{}^{\backslash}m_{\backslash}][{}^{\backslash}\omega_{r\backslash}^2]$.

The matrix formulation for the frequency response function matrix for this undamped case becomes:

(a.1.80) $$[H]=[\psi]\left[[{}^{\backslash}\omega_{r\backslash}^2]-\omega^2[{}^{\backslash}I_{\backslash}]\right]^{-1}[{}^{\backslash}m_{\backslash}]^{-1}[\psi]^t.$$

This expression compares to equation a.1.56.

For proportionally damped systems equation a.1.81 defines the *modal damping matrix*:

(a.1.81) $$[\psi]^t[C][\psi]=[{}^{\backslash}c_{\backslash}],$$

yielding the following set of decoupled equations (after the transformation $\{X\}=[\psi]\{q\}$):

(a.1.82) $$\left(p^2[{}^{\backslash}m_{\backslash}]+p[{}^{\backslash}c_{\backslash}]+[{}^{\backslash}k_{\backslash}]\right)\{q\}=[\psi]^t\{F\}.$$

Modal mass, stiffness and damping depend on the modal scaling.

Considering equation a.1.82 at a particular resonance $\lambda_r=\sigma_r+j\omega_r$ yields:

(a.1.83) $$(\sigma_r+j\omega_r)^2 m_r+(\sigma_r+j\omega_r)c_r+k_r=0$$

The real part yields: $2\sigma_r\omega_r m_r+c_r\omega_r=0$ or $c_r=-2\sigma_r m_r$.

The imaginary part then yields: $-(\sigma_r^2+\omega_r^2)m_r+k_r=0$.

The matrix formulation for the frequency response function matrix for the proportionally damped case becomes:

(a.1.84) $$[H(j\omega)]=[\psi]\left[[{}^{\backslash}\sigma_r^2+\omega_{r\backslash}^2]-\omega^2[{}^{\backslash}I_{\backslash}]-2j\omega[{}^{\backslash}\sigma_{r\backslash}]\right]^{-1}[{}^{\backslash}m_{\backslash}]^{-1}[\psi]^t.$$

This expression compares to equation a.1.61.

Interpretation:

The modal mass, modal stiffness and modal damping for the proportionally damped case, and the modal *a* and modal *b* for the general viscous damped case, are in fact only scaling factors for the mode shapes. However, for the proportionally damped case, modal mass, stiffness and damping represent the mass, stiffness and damping of the sungle degree of freedom systems, after decoupling of the equations by transformation to modal coordinates (see eq.a.1.82). The modal *a* and *b* are more abstract notions.

A.1.2.8. Modal vector scaling.

As stated before (section A.1.2.3), equations a.1.35 and a.1.45 show that the residues $[A]_r$ are absolute quantities and that the modal vectors $\{\psi\}_r$ are scaled vectors, dependent on the scaling factors Q_r. They are related by equation a.1.45:

(a.1.45) $$[A]_r=Q_r\{\psi\}_r\{\psi\}_r^t.$$

For element *pq* of the residue matrix $[A]_r$ (i.e. response degree of freedom *p*, input degree of freedom *q*):

(a.1.85) $$A_{pqr}=Q_r\psi_{pr}\psi_{qr}$$

Comparing equations a.1.76 and a.1.46 for the transfer function matrix for a system with general viscous damping proves that the scaling factor Q_r equals the inverse of the corresponsing modal a_r or:

(a.1.86) $$[{}^{\backslash}Q_{\backslash}]=[{}^{\backslash}a_{\backslash}]^{-1}.$$

In case of an experimental modal analysis mass, stiffness and damping matrices ([*M*], [*K*], [*C*]) are unknown. Hence, the modal a_r-matrix, $[{}^{\backslash}a_{\backslash}]$, can not be calculated according to equation a.1.68. However, measuring the direct frequency response function, i.e. the frequency response function between the output and the input in the same measurement degree of freedom (location and orientation), determines the residue

A_{qqr} for mode r. Combining equations a.1.85 and a.1.86 shows how the modal a_r for mode r can be found from frequency response function measurements:

(a.1.87) $$a_r = \frac{\psi_{qr}\psi_{qr}}{A_{qqr}}.$$

Following scaling schemes are common:

Unit modal a:

(a.1.88) $$a_r = 1, Q_r = 1, \psi_{qr} = \sqrt{A_{qqr}}, \psi_{pr} = \frac{A_{pqr}}{\sqrt{A_{qqr}}}.$$

Unity modal coefficient:

Assume that the i-th component of mode r, ψ_{ir}, must be unity:

(a.1.89) $$\psi_{ir} = 1, \psi_{pr} = \frac{A_{pqr}}{A_{iqr}}, Q_r = \frac{A_{pqr}}{\psi_{pr}\psi_{qr}} = \frac{1}{a_r}.$$

Unity modal vector length:

(a.1.90) $$\left\|\{A_q\}_r\right\| = \sqrt{\sum_{i=1}^{N} A_{iqr}A_{iqr}^{*}}, \psi_{pr} = \frac{A_{pqr}}{\left\|\{A_q\}_r\right\|}, Q_r = \frac{A_{pqr}}{\psi_{pr}\psi_{qr}} = \frac{1}{a_r}$$

For undamped systems equations a.1.56 and a.1.80,

(a.1.56) $$[H(j\omega)] = [\psi]\left[[{}^{\backslash}\omega_{r}^{2}{}_{\backslash}] - \omega^2[{}^{\backslash}I_{\backslash}]\right]^{-1}[{}^{\backslash}j2\omega_r Q_{\backslash}][\psi]^t$$

(a.1.80) $$[H] = [\psi]\left[[{}^{\backslash}\omega_{r}^{2}{}_{\backslash}] - \omega^2[{}^{\backslash}I_{\backslash}]\right]^{-1}[{}^{\backslash}m_{\backslash}]^{-1}[\psi]^t,$$

show that:

(a.1.91) $$[{}^{\backslash}m_{\backslash}]^{-1} = [{}^{\backslash}j2\omega_r Q_{\backslash}] \text{ or}$$
$$Q_r = \frac{1}{j2m_r\omega_r}$$

Hence, the *unity modal mass* scaling scheme is:

(a.1.92) $$m_r = 1, Q_r = \frac{1}{j2\omega_r}, \psi_{qr} = \sqrt{\frac{A_{qqr}}{Q_r}}, \psi_{pr} = \frac{A_{pqr}}{Q_r\psi_{qr}}.$$

Comparing equations a.1.61 and a.1.84 for the proportionally damped case defines the same scaling scheme. Unity modal mass scaling is very often used, even in non-proportionally damped cases.

A.1.2.9. Analytical and experimental approach

The analytical approach to modal analysis starts with the estimation of the mass, stiffness and damping distribution of the structure by means of a mass, stiffness and damping matrix ([*M*], [*K*], [*C*]). These define the eigenvalue problem as stated by equation a.1.29:

(a.1.29) $$\left(p\begin{bmatrix}[0] & [M]\\ [M] & [C]\end{bmatrix} + \begin{bmatrix}-[M] & [0]\\ [0] & [K]\end{bmatrix}\right)\{Y\} = \{0\}.$$

The eigenvalues are the system poles $\lambda_r = \sigma_r + j\omega_r$, containing the damping factors and damped natural frequencies. The eigenvectors are related to the modal vectors $\{\psi\}_r$ (eq. a.1.32):

(a.1.32) $$[\Phi] = \begin{bmatrix}\lambda_1\{\psi\}_1 & \cdots & \lambda_N\{\psi\}_N & \lambda_1^{*}\{\psi\}_1^{*} & \cdots & \lambda_N^{*}\{\psi\}_N^{*}\\ \{\psi\}_1 & \cdots & \{\psi\}_N & \{\psi\}_1^{*} & \cdots & \{\psi\}_N^{*}\end{bmatrix}.$$

The eigenvectors and the system matrices determine the modal a matrix (eq. a.1.68):

(a.1.68) $$\begin{bmatrix}\lambda_1\{\psi\}_1^t & \{\psi\}_1^t\\ \vdots & \vdots\\ \lambda_N^{*}\{\psi\}_N^{*t} & \{\psi\}_N^{*t}\end{bmatrix}\begin{bmatrix}[0] & [M]\\ [M] & [C]\end{bmatrix}\begin{bmatrix}\lambda_1\{\psi\}_1 & \cdots & \lambda_N^{*}\{\psi\}_N^{*}\\ \{\psi\}_1 & \cdots & \{\psi\}_N^{*}\end{bmatrix} = [{}^{\backslash}a_{\backslash}].$$

Equation a.1.86, $[{}^{\backslash}Q_{\backslash}] = [{}^{\backslash}a_{\backslash}]^{-1}$, yields the appropriate scaling factors to construct the frequency response function matrix in terms of the modal parameters (eq.a.1.48):

(a.1.48) $$[H(j\omega)] = \sum_{r=1}^{N}\left(\frac{Q_r\{\psi\}_r\{\psi\}_r^t}{(j\omega - \lambda_r)} + \frac{Q_r^{*}\{\psi\}_r^{*}\{\psi\}_r^{*t}}{(j\omega - \lambda_r^{*})}\right).$$

The experimental approach starts with measuring (parts of) this frequency response function matrix. In the next step, parameter estimation techniques (see chapter A.3) use equation a.1.48 (its time domain equivalent, the impulse response function, or related equations, such as the direct time responses) to estimate the modal parameters, λ_r, $\{\psi\}_r$, and the scaling factors, Q_r, according to one of the scaling schemes mentioned in section A.1.2.8. Equation a.1.86, $[{}^{\backslash}Q_{\backslash}] = [{}^{\backslash}a_{\backslash}]^{-1}$, yields the corresponding values for the modal a matrix. Since the experimental modal data base is generally incomplete (the number of degrees of freedom exceeds largely the number of estimated modes) the system mass, stiffness and damping matrix can not be estimated correctly from these modal data.

A.1.3. SINGLE DEGREE OF FREEDOM SYSTEM: EXAMPLE

Consider a single degree of freedom system with:

$$M = 2\,kg,\ C = 4\frac{N}{m/s},\ K = 5000\frac{N}{m}.$$

The system equation in the Laplace domain is:

$$\left(Mp^2 + Cp + K\right)X(p) = F(p)$$
$$\left(2p^2 + 4p + 5000\right)X(p) = F(p),$$

where $Z(p) = (2p^2 + 4p + 5000)$ is the dynamic stiffness.

The transfer function is the inverse of the dynamic stiffness,

$$H(p) = \frac{1/M}{p^2 + (C/M)p + (K/M)} = \frac{1/2}{p^2 + 2p + 2500}.$$

The system poles, i.e. the roots of the characteristic equation $(p^2 + 2p + 2500)$, are:

$$\lambda_{1,2} = -(C/2M) \pm \sqrt{(C/2M)^2 - (K/M)}$$
$$\lambda_{1,2} = -1 \pm \sqrt{1 - 2500} = -1 \pm j49.9900\,rad/s.$$

The undamped natural frequency $\Omega_1 = \sqrt{K/M} = 50\,rad/s = 7.9577\,Hz$.

The critical damping $C_c = 2M\sqrt{K/M} = 200\frac{N}{m/s}$ and the damping ratio $\zeta_1 = C/C_c = 0.02$ or 2%.

The residue $A_1 = \frac{1/M}{j2\omega_1} = -j5.001\times10^{-3}\,s/kg$. Hence, the partial fraction formulation of the transfer function is:

$$H(p) = \frac{A_1}{(p-\lambda_1)} + \frac{A_1^*}{(p-\lambda_1^*)} = \frac{-j5.001\times10^{-3}}{(p-(-1+j49.9900))} + \frac{j5.001\times10^{-3}}{(p-(-1-j49.9900))}.$$

A.1.4. MULTIPLE DEGREE OF FREEDOM SYSTEM: EXAMPLE

A.1.4.1. General viscous damping

Consider a two degree of freedom system, as depicted in figure a.1.11, where the masses, dampers and stiffnesses have following values:

$$M_1 = M_2 = 2\,kg,$$
$$C_1 = 3\frac{N}{m/s},\ C_2 = 1\frac{N}{m/s}, C_3 = 4\frac{N}{m/s},$$
$$K_1 = 4000\frac{N}{m},\ K_2 = 2000\frac{N}{m},\ K_3 = 4000\frac{N}{m}.$$

These values yield following system equation in the Laplace domain:

$$[Z(p)]\{X(p)\} = \left(p^2\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} + p\begin{bmatrix}4 & -1\\-1 & 5\end{bmatrix} + \begin{bmatrix}6000 & -2000\\-2000 & 6000\end{bmatrix}\right)\{X(p)\} = \{F(p)\}$$

The transfer function matrix is:

$$[H(p)] = [Z(p)]^{-1} = \left[p^2\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} + p\begin{bmatrix}4 & -1\\-1 & 5\end{bmatrix} + \begin{bmatrix}6000 & -2000\\-2000 & 6000\end{bmatrix}\right]^{-1} \quad \text{or}$$

$$[H(p)] = \frac{adj([Z(p)])}{|Z(p)|} = \frac{\begin{bmatrix}2p^2+5p+6000 & p+2000\\ p+2000 & 2p^2+4p+6000\end{bmatrix}}{(2p^2+4p+6000)(2p^2+5p+6000)-(p+2000)^2}$$

The system poles and corresponding modal vectors are the eigenvalues and eigenvectors of the following eigenvalue problem (see eq. a.1.29, $(p[A]+[B])\{Y\} = \{0\}$):

$$\left(p\begin{bmatrix}0 & 0 & 2 & 0\\0 & 0 & 0 & 2\\2 & 0 & 4 & -1\\0 & 2 & -1 & 5\end{bmatrix} + \begin{bmatrix}-2 & 0 & 0 & 0\\0 & -2 & 0 & 0\\0 & 0 & 6000 & -2000\\0 & 0 & -2000 & 6000\end{bmatrix}\right)\{Y\} = \{0\}$$

This results in:

$$\lambda_1 = -0.87501 + 44.7135j = 44.722^{\angle 91.246^\circ} rad/s$$

$$\begin{Bmatrix} \lambda_1\{\psi\}_1 \\ \{\psi\}_1 \end{Bmatrix} = \begin{Bmatrix} 4.0321\times10^{-3} + 7.0693\times10^{-1} \\ 1.1937\times10^{-2} + 7.0682\times10^{-1} \\ 1.5802\times10^{-2} - 3.9942\times10^{-4} \\ 1.5796\times10^{-2} - 5.7609\times10^{-4} \end{Bmatrix} = \begin{Bmatrix} 7.0694\times10^{-1\ \angle 89.673^\circ} \\ 7.0692\times10^{-1\ \angle 89.032^\circ} \\ 1.5807\times10^{-2\ \angle -1.4479^\circ} \\ 1.5807\times10^{-2\ \angle -2.0886^\circ} \end{Bmatrix} \quad \text{and}$$

$$\lambda_2 = -1.3750 + 63.2296j = 63.245^{\angle 91.121^\circ} rad/s$$

$$\begin{Bmatrix} \lambda_2\{\psi\}_2 \\ \{\psi\}_2 \end{Bmatrix} = \begin{Bmatrix} -9.1825\times10^{-2} + 7.0097\times10^{-1} \\ 1.0291\times10^{-1} + 6.9954\times10^{-1} \\ 1.1112\times10^{-2} + 1.2106\times10^{-3} \\ -1.1094\times10^{-2} - 1.3864\times10^{-3} \end{Bmatrix} = \begin{Bmatrix} 7.0696\times10^{-1\ \angle 97.461^\circ} \\ 7.0707\times10^{-1\ \angle -81.631^\circ} \\ 1.1178\times10^{-2\ \angle 6.2173^\circ} \\ 1.1180\times10^{-2\ \angle -172.88^\circ} \end{Bmatrix}$$

and the complex conjugates: $\lambda_1^*, \begin{Bmatrix} \lambda_1^*\{\psi\}_1^* \\ \{\psi\}_1^* \end{Bmatrix}; \quad \lambda_2^*, \begin{Bmatrix} \lambda_2^*\{\psi\}_2^* \\ \{\psi\}_2^* \end{Bmatrix}$.

Equation a.1.35 expresses the transfer function matrix in terms of residues and system poles:

(a.1.35) $$[H(p)] = \sum_{r=1}^{N}\left(\frac{[A]_r}{(p-\lambda_r)} + \frac{[A]_r^*}{(p-\lambda_r^*)}\right)$$

where:

$$[A]_r = \frac{adj([Z(\lambda_r)])}{\prod_{s=1, s\neq r}^{2N} E(\lambda_r - \lambda_s)} \quad \text{or}$$

$$[A]_r = P_r\, adj([Z(\lambda_r)]).$$

In this example:

$$[A]_1 = (-3.1266\times10^{-8} - 1.3978\times10^{-6}j)\begin{bmatrix} 1998.6+67.1j & 1999.1+44.7j \\ 1999.1+44.7j & 1999.4+22.4j \end{bmatrix}$$

$$[A]_1 = \begin{bmatrix} 3.1263\times10^{-5} - 2.7958\times10^{-3}j & -2.9326\times10^{-9} - 2.7958\times10^{-3}j \\ -2.9326\times10^{-9} - 2.7958\times10^{-3}j & -3.1267\times10^{-5} - 2.7958\times10^{-3}j \end{bmatrix} \quad \text{and}$$

$$[A]_2 = (-3.1266\times10^{-8} + 9.8824\times10^{-7}j)\begin{bmatrix} 1999.1-31.6j & 1998.6+63.2j \\ 1998.6+63.2j & -1997.7-94.8j \end{bmatrix}$$

$$[A]_2 = \begin{bmatrix} -3.1263\times10^{-5} - 1.9766\times10^{-3}j & 2.9326\times10^{-9} + 1.9771\times10^{-3}j \\ 2.9326\times10^{-9} + 1.9771\times10^{-3}j & 3.1267\times10^{-5} - 1.9772\times10^{-3}j \end{bmatrix}$$

and the complex conjugates $[A]_1^*$ and $[A]_2^*$.

Equation a.1.45, $[A]_r = Q_r\{\psi\}_r\{\psi\}_r^t$, defines the scaling constant Q_r for the modal vector $\{\psi\}_r$. In this example equation a.1.45 becomes:

$$[A]_1 = (0.69019 - 11.168j)\begin{bmatrix} 2.4956\times10^{-4} - 1.2624\times10^{-5}j & 2.4939\times10^{-4} - 1.5413\times10^{-5}j \\ 2.4939\times10^{-4} - 1.5413\times10^{-5}j & 2.4920\times10^{-4} - 1.8200\times10^{-5}j \end{bmatrix}$$

$$[A]_2 = (-3.6504 - 15.393j)\begin{bmatrix} 1.2202\times10^{-4} + 2.6905\times10^{-5}j & -1.2160\times10^{-4} - 2.8836\times10^{-5}j \\ -1.2160\times10^{-4} - 2.8836\times10^{-5}j & 1.2115\times10^{-4} + 3.0760\times10^{-5}j \end{bmatrix}$$

and the complex conjugates $[A]_1^*$ and $[A]_2^*$.

The orthogonality conditions (a.1.68 and a.1.69) for this example are:

$$[\Phi]^t[A][\Phi] = 10^{-2}\begin{bmatrix} .55127+8.9201j & 0 & 0 & 0 \\ 0 & -1.4585+6.1504j & 0 & 0 \\ 0 & 0 & .55127-8.9201j & 0 \\ 0 & 0 & 0 & -1.4585-6.1504j \end{bmatrix}$$

$$[\Phi]^t[B][\Phi] = \begin{bmatrix} 3.9933-.16844j & 0 & 0 & 0 \\ 0 & 3.8688+1.0068j & 0 & 0 \\ 0 & 0 & 3.9933+.16844j & 0 \\ 0 & 0 & 0 & 3.8688-1.0068j \end{bmatrix}$$

The reader can check that these values comply with equations a.1.86, $[{}^{\backslash}Q_{\backslash}] = [{}^{\backslash}a_{\backslash}]^{-1}$, and a.1.74, $[{}^{\backslash}b_{\backslash}] = -[{}^{\backslash}a_{\backslash}][{}^{\backslash}\Lambda_{\backslash}]$.

A.1.4.2. Proportional viscous damping

A two degree of freedom system, as depicted in figure a.1.11, with mass, damping and stiffness values:

$$M_1 = M_2 = 2kg,$$
$$C_1 = 3\frac{N}{m/s},\ C_2 = 2\frac{N}{m/s}, C_3 = 3\frac{N}{m/s},$$
$$K_1 = 4000\frac{N}{m},\ K_2 = 2000\frac{N}{m},\ K_3 = 4000\frac{N}{m},$$

is a proportionally damped system. The coefficients α and β of equation a.1.57, $[C] = \alpha[M] + \beta[K]$, are:

$$\alpha = -\frac{1}{2}\ 1/s,\ \beta = \frac{1}{1000}\ s.$$

The corresponding eigenvalues and eigenvectors are:

$$\lambda_1 = -0.7500 + 44.7151j = 44.721^{\angle 90.961^\circ} rad/s$$

$$\begin{Bmatrix} \lambda_1\{\psi\}_1 \\ \{\psi\}_1 \end{Bmatrix} = \begin{Bmatrix} -2.1660\times10^{-1} + 6.7293\times10^{-1} \\ -2.1660\times10^{-1} + 6.7293\times10^{-1} \\ 1.5126\times10^{-2} + 4.5903\times10^{-3} \\ 1.5126\times10^{-2} + 4.5903\times10^{-3} \end{Bmatrix} = \begin{Bmatrix} 7.0693\times10^{-1\ \angle 107.84^\circ} \\ 7.0693\times10^{-1\ \angle 107.84^\circ} \\ 1.5807\times10^{-2\ \angle 16.881^\circ} \\ 1.5807\times10^{-2\ \angle 16.881^\circ} \end{Bmatrix} \quad \text{and}$$

$$\lambda_2 = -1.7500 + 63.2213j = 63.246^{\angle 91.586^\circ} rad/s$$

$$\begin{Bmatrix} \lambda_2\{\psi\}_2 \\ \{\psi\}_2 \end{Bmatrix} = \begin{Bmatrix} 2.0629\times10^{-1} + 6.7626\times10^{-1} \\ -2.0629\times10^{-1} - 6.7626\times10^{-1} \\ 1.0598\times10^{-1} - 3.5563\times10^{-3} \\ -1.0598\times10^{-1} + 3.5563\times10^{-3} \end{Bmatrix} = \begin{Bmatrix} 7.0702\times10^{-1\ \angle 73.063^\circ} \\ 7.0702\times10^{-1\ \angle -106.96^\circ} \\ 1.1179\times10^{-2\ \angle -18.549^\circ} \\ 1.1179\times10^{-2\ \angle 161.45^\circ} \end{Bmatrix}$$

and the complex conjugates: $\lambda_1^*, \begin{Bmatrix} \lambda_1^*\{\psi\}_1^* \\ \{\psi\}_1^* \end{Bmatrix}; \ \lambda_2^*, \begin{Bmatrix} \lambda_2^*\{\psi\}_2^* \\ \{\psi\}_2^* \end{Bmatrix}$.

As stated in section A.1.2.6. the modal vectors $\{\psi\}_r$ are normal modal vectors: the phases of the elements are equal (e.g. mode 1) or differ exactly 180° (e.g.mode 2). If they are properly rescaled they equal the modal vectors for the undamped case (section A.1.4.3). The system poles are complex. Their modulus equals the modulus of the system poles of the undamped system (section A.1.4.3). This is not true for the general viscous damping case.

Other quantities as the residues, modal a and modal b, modal scaling factors can be calculated similarly as for the general viscous damping case.

A.1.4.3. No damping

A two degree of freedom system, as depicted in figure a.1.11, without damping and with mass and stiffness values:

$$M_1 = M_2 = 2kg,$$
$$K_1 = 4000\frac{N}{m},\ K_2 = 2000\frac{N}{m},\ K_3 = 4000\frac{N}{m},$$

yields an eigenvalue problem as stated by equation a.1.55 ($(p^2[M]+[K])\{X\} = \{0\}$). The purely imaginary system poles, and corresponding modal vectors are:

$$\lambda_1 = 44.721j\ rad/s,\ \{\psi\}_1 = \begin{Bmatrix} 0.7071 \\ 0.7071 \end{Bmatrix},$$
$$\lambda_2 = 63.246j\ rad/s,\ \{\psi\}_2 = \begin{Bmatrix} 0.7071 \\ -0.7071 \end{Bmatrix}$$

and the complex conjugates $\lambda_1^*, \{\psi\}_1^*;\ \lambda_2^*, \{\psi\}_2^*$.

As stated before, the modal vectors and the modulus of the system poles are identical to these of the proportionally damped case.

Equations a.1.77 and a.1.78 ($[\psi]^t[M][\psi] = [{}^{\backslash}m_{\backslash}]$ and $[\psi]^t[K][\psi] = [{}^{\backslash}k_{\backslash}]$) define the modal masses and stiffnesses:

$$\begin{bmatrix} .7071 & .7071 \\ .7071 & -.7071 \end{bmatrix}^t \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} .7071 & .7071 \\ .7071 & -.7071 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\begin{bmatrix} .7071 & .7071 \\ .7071 & -.7071 \end{bmatrix}^t \begin{bmatrix} 6000 & -2000 \\ -2000 & 6000 \end{bmatrix} \begin{bmatrix} .7071 & .7071 \\ .7071 & -.7071 \end{bmatrix} = \begin{bmatrix} 4000 & 0 \\ 0 & 8000 \end{bmatrix}.$$

According to equations a.1.80 and a.1.56 the frequency response function matrix is:

$$[H(j\omega)] = \sum_{r=1}^{N} \frac{\{\psi\}_r\{\psi\}_r^t}{m_r(\omega_r^2 - \omega^2)} = \sum_{r=1}^{N} \frac{j2\omega_r Q_r\{\psi\}_r\{\psi\}_r^t}{(\omega_r^2 - \omega^2)},$$
$$[H(j\omega)] = \frac{0.5\begin{Bmatrix} .7071 \\ .7071 \end{Bmatrix}\begin{Bmatrix} .7071 \\ .7071 \end{Bmatrix}^t}{2000 - \omega^2} + \frac{0.5\begin{Bmatrix} .7071 \\ -.7071 \end{Bmatrix}\begin{Bmatrix} .7071 \\ -.7071 \end{Bmatrix}^t}{4000 - \omega^2}.$$

The corresponding modal scale factors, according to equation a.1.91, are:

$$Q_1 = -5.5902\times10^{-3}j \quad \text{and} \quad Q_2 = -3.9528\times10^{-3}j.$$

A.1.5. CONCLUSIONS

This chapter covered the basic theory of modal analysis. It showed that the dynamics of a structure are completely described by the modal parameters. The chapter explained that these modal parameters can be derived from the knowledge of the mass, stiffness and damping matrices of a model of the structure (the analytical approach) or from the measurement of frequency response functions on the structure (the experimental approach). A small numerical example illustrated the theory.